Automatically Synthesised Selection Algorithm

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Theorem 1. $V_4(24) \leq 34$

Proof. Algorithm 1 presents a method for computing the 4 -th largest of 24 elements. In the worst-case, the number of comparisons is equal to 34. □

Algorithm 1. *Let KEYS be a totally ordered set, |KEYS| =24.*

1. *Partition the set KEYS into disjoint subsets |K1|=16, |K2|=8. Determine the maxima of K1, K2 by setting-up balanced tournaments. The resulting poset is isomorphic to the poset as shown in Figure 1. For setting-up the balanced tournaments 22 comparisons are needed.*

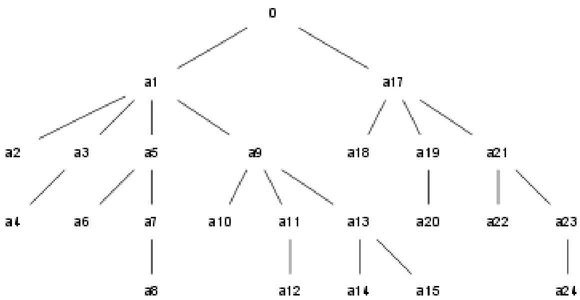

Figure 1: Balanced Tournaments of Set Partition

2. (a) *Perform the comparisons in accordance with the decision graph in Figure 2*[1]

[1] The nodes in the decision graph represent:

i. Comparisons X<Y, shown as circles.Non-filled arrows represent the case X>Y and filled arrows refer to the case X<Y.

ii. Subgraph place holders presented by diamonds, which correspond to solutions which can be obtained as instances from theorems and algorithms known from published literature.

iii. Place holders for calls to isomorphism functions, which are denoted by double circles. There are two type of function calls. Firstly, solutions for the isomorphic subcases are part of the presented proof graph and.are being refer<enced. Secondly, in some situations more detailed explanations for the isomorphic subcases have been generated. The referenced cases are indicated by circles.

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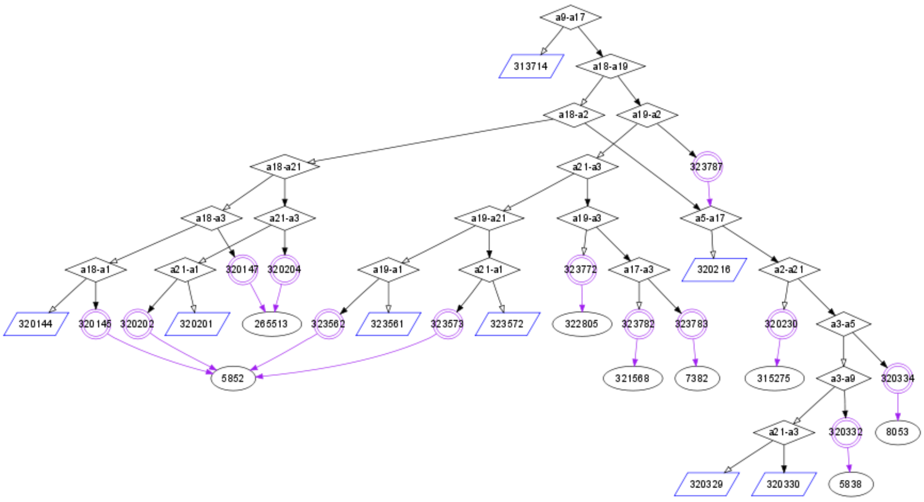

Figure 2: decision graph

(b) *Further steps correspond up to isomorphism with solutions which are retrieved from the case base.*
Node 5838 refers to Case 19.
Node 5852 refers to Case 18.
Node 7382 refers to Case 4.
Node 8053 refers to Case 5.
Node 265513 refers to Case 7.
Node 315275 refers to Case 10.
Node 321568 refers to Case 15.
Node 322805 refers to Case 17.

(c) *Solutions for the subproblems associated with the nodes* 313714,320144,320201,320216,320329,320330,323561,323572 *may be obtained as instances of the Algorithms* Aigner (1982) Hadian and Sobel (1969) Kislitsyn (1964) .

Case 1. *Let P be the partially ordered set as visualised in Figure 21. The 3-rd largest element of P can be computed by at most 5 comparisons.*

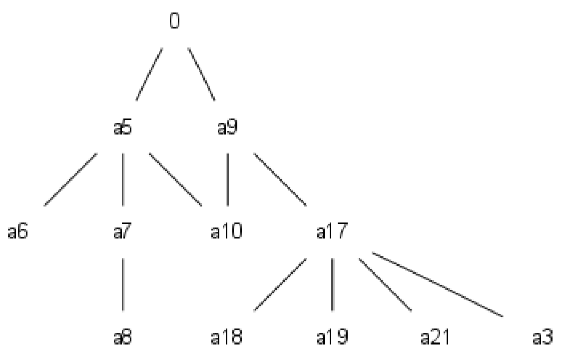

Figure 3: Poset P

Proof. Compare a17 and a5.
if *a17 > a5* **then**
Selecting the 1st largest element takes at most f1(5)=4 further comparisons Knuth (1973).

else
Selecting the 2nd largest element takes at most f2(0,1,2)=4 further comparisons Kislitsyn (1964).
end

□

Case 2. *Let P be the partially ordered set as visualised in Figure 20. The 3-rd largest element of P can be computed by at most 6 comparisons.*

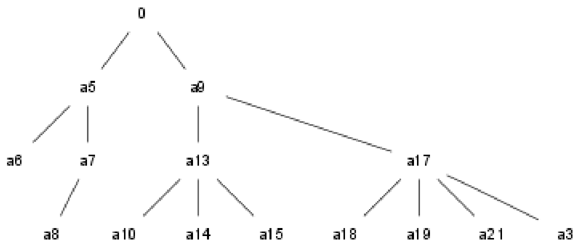

Figure 4: Poset P

Proof. Compare a13 and a5.
if *a13 > a5* **then**
Selecting the 2nd largest element takes at most f2(4,4)=5 further comparisons Kislitsyn (1964).
else
The subsolution corresponds via the following isomorphism to Case 19.

$$\phi : [a9-a9, a17-a17, a18-a18, a19-a19, a21-a21, a3-a3, a13-a10, a5-a5, a6-a6, a7-a7, a8-a8]$$

end

□

Case 3. *Let P be the partially ordered set as visualised in Figure 21. The 3-rd largest element of P can be computed by at most 5 comparisons.*

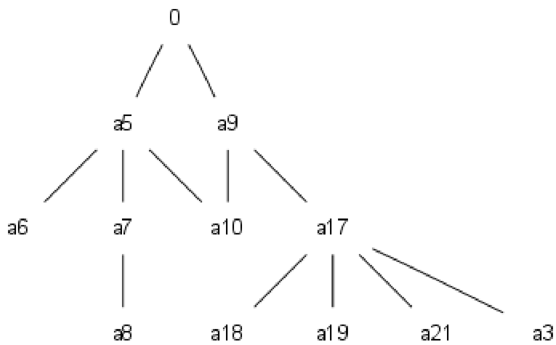

Figure 5: Poset P

Proof. Compare a17 and a5.
if *a17 > a5* **then**
Selecting the 1st largest element takes at most f1(5)=4 further comparisons Knuth (1973).
else
Selecting the 2nd largest element takes at most f2(0,1,2)=4 further comparisons Kislitsyn (1964).
end

□

Case 4. *Let P be the partially ordered set as visualised in Figure 6. The 3-rd largest element of P can be computed by at most 6 comparisons.*

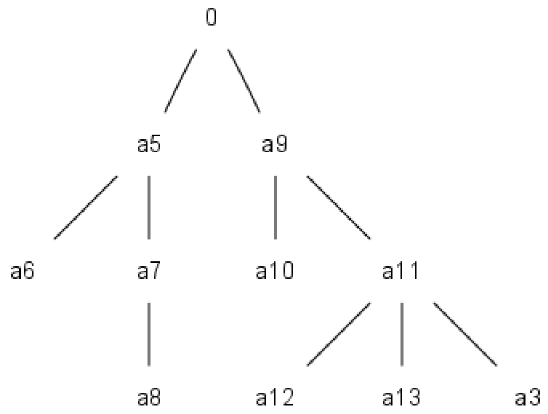

Figure 6: Poset P

Proof. Compare a9 and a5.
if *a9 > a5* **then**
Selecting the 2nd largest element takes at most f2(0,2,3)=5 further comparisons Kislitsyn (1964).
else
Selecting the 2nd largest element takes at most f2(0,1,2)=4 further comparisons Kislitsyn (1964).
end

□

Case 5. *Let P be the partially ordered set as visualised in Figure 7. The 3-rd largest element of P can be computed by at most 6 comparisons.*

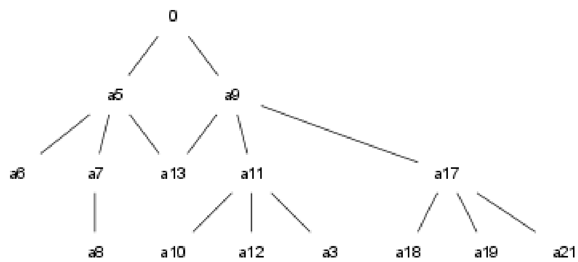

Figure 7: Poset P

Proof. Compare a11 and a17.
if *a11 > a17* **then**
The subsolution corresponds via the following isomorphism to Case 19.

$$\phi : [a9-a9, a11-a17, a10-a18, a12-a19, a17-a21, a3-a3, a13-a10, a5-a5, a6-a6, a7-a7, a8-a8]$$

else
The subsolution corresponds via the following isomorphism to Case 19.

$$\phi : [a9-a9, a17-a17, a11-a18, a18-a19, a19-a21, a21-a3, a13-a10, a5-a5, a6-a6, a7-a7, a8-a8]$$

end

□

Case 6. *Let P be the partially ordered set as visualised in Figure 21. The 3-rd largest element of P can be computed by at most 5 comparisons.*

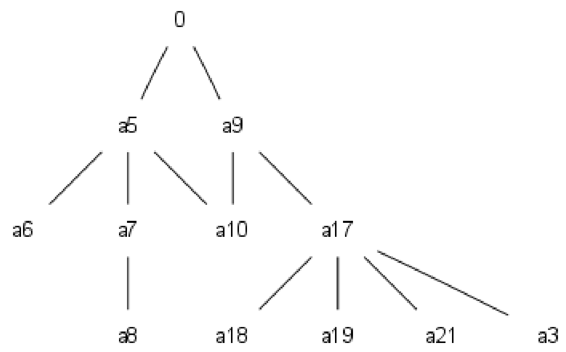

Figure 8: Poset P

Proof. Compare a17 and a5.
if *a17 > a5* **then**
Selecting the 1st largest element takes at most f1(5)=4 further comparisons Knuth (1973).
else
Selecting the 2nd largest element takes at most f2(0,1,2)=4 further comparisons Kislitsyn (1964).
end

□

Case 7. *Let P be the partially ordered set as visualised in Figure 9. The 3-rd largest element of P can be computed by at most 7 comparisons.*

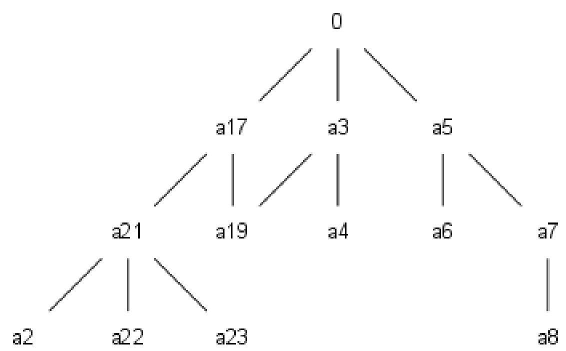

Figure 9: Poset P

Proof. Compare a17 and a3.
if *a17 > a3* **then**
The subsolution corresponds via the following isomorphism to Case 8.

$$\phi : [a5-a5, a6-a6, a7-a7, a8-a8, a17-a9, a3-a10, a19-a11, a4-a3, a21-a17, a2-a18, a22-a19, a23-a21]$$

else
The subsolution corresponds via the following isomorphism to Case 9.

$$\phi : [a5-a9, a6-a10, a7-a11, a8-a12, a3-a5, a4-a6, a17-a7, a19-a3, a21-a8]$$

end

□

Case 8. *Let P be the partially ordered set as visualised in Figure 10. The 3-rd largest element of P can be computed by at most 6 comparisons.*

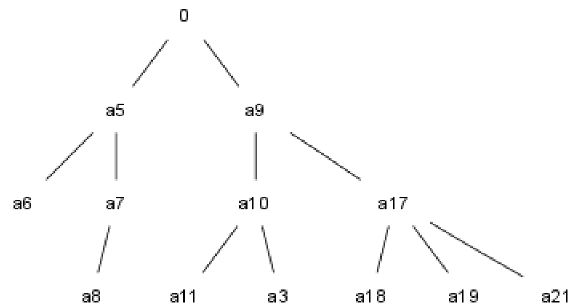

Figure 10: Poset P

Proof. Compare a9 and a5.
if *a9 > a5* **then**
Selecting the 2nd largest element takes at most f2(2,2,3)=5 further comparisons Kislitsyn (1964).
else
Selecting the 2nd largest element takes at most f2(0,1,2)=4 further comparisons Kislitsyn (1964).
end

□

Case 9. *Let P be the partially ordered set as visualised in Figure 11. The 3-rd largest element of P can be computed by at most 6 comparisons.*

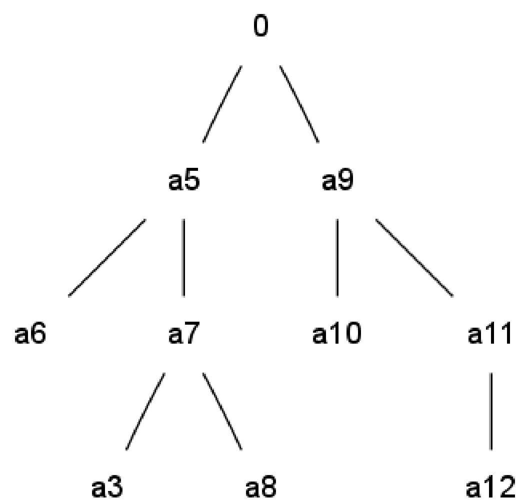

Figure 11: Poset P

Proof. Compare a5 and a9.
if *a5 > a9* **then**
Selecting the 2nd largest element takes at most f2(0,2,2)=5 further comparisons Kislitsyn (1964).
else
Selecting the 2nd largest element takes at most f2(0,1,2)=4 further comparisons Kislitsyn (1964).

end

□

Case 10. *Let P be the partially ordered set as visualised in Figure 12. The 3-rd largest element of P can be computed by at most 7 comparisons.*

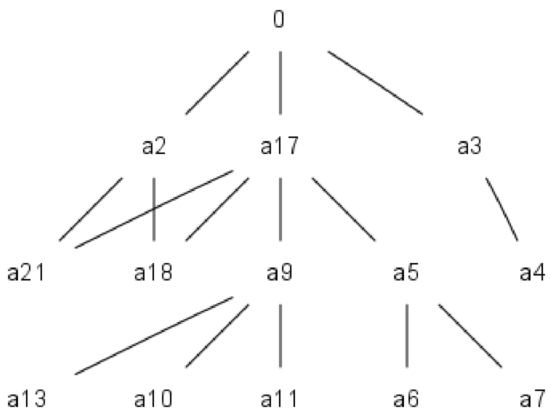

Figure 12: Poset P

Proof. Compare a2 and a3.
if *a2 > a3* **then**
The subsolution corresponds via the following isomorphism to Case 12.

$\phi : [a2-a18, a3-a19, a4-a20, a18-a2, a21-a3, a17-a1, a5-a5, a6-a6, a7-a7, a9-a9, a10-a10, a11-a11, a13-a13$

else
The subsolution corresponds via the following isomorphism to Case 11.

$\phi : [a3-a3, a2-a2, a4-a4, a17-a9, a5-a10, a6-a11, a7-a5, a9-a17, a10-a18, a11-a19, a13-a21]$

end

□

Case 11. *Let P be the partially ordered set as visualised in Figure 13. The 3-rd largest element of P can be computed by at most 6 comparisons.*

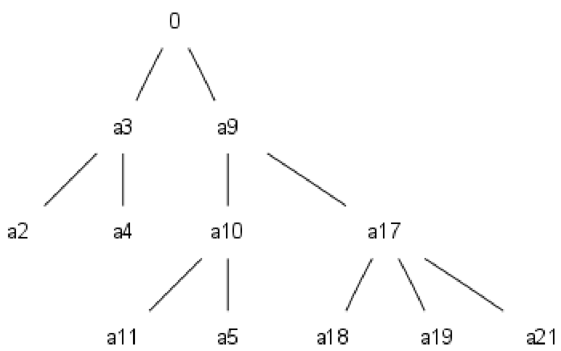

Figure 13: Poset P

Proof. Compare a9 and a3.
if *a9 > a3* **then**
Selecting the 2nd largest element takes at most f2(2,2,3)=5 further comparisons Kislitsyn (1964).
else
Selecting the 2nd largest element takes at most f2(0,0,2)=4 further comparisons Kislitsyn (1964).
end

□

Case 12. *Let P be the partially ordered set as visualised in Figure 14. The 3-rd largest element of P can be computed by at most 6 comparisons.*

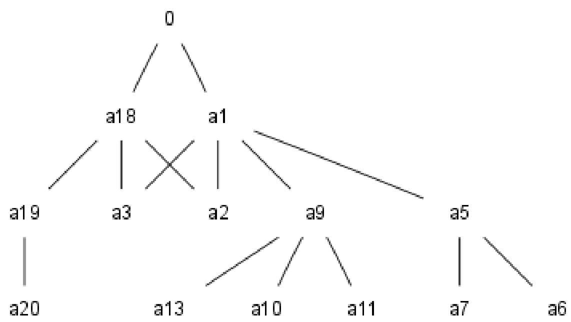

Figure 14: Poset P

Proof. Compare a9 and a18.
if *a9 > a18* **then**
Selecting the 2nd largest element takes at most f2(2,4)=5 further comparisons Kislitsyn (1964).
else
The subsolution corresponds via the following isomorphism to Case 13.

$$\phi : [a18-a3, a19-a4, a20-a5, a2-a10, a3-a13, a9-a17, a1-a9, a5-a11, a6-a12, a7-a2]$$

end

□

Case 13. *Let P be the partially ordered set as visualised in Figure 15. The 3-rd largest element of P can be computed by at most 5 comparisons.*

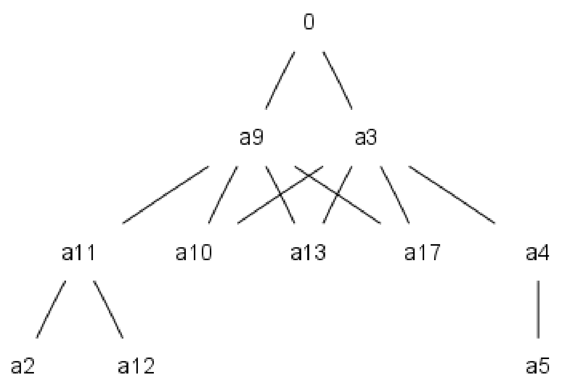

Figure 15: Poset P

Proof. Compare a10 and a11.
if *a10 > a11* **then**
Selecting the 2nd largest element takes at most f2(1,3)=4 further comparisons Kislitsyn (1964).
else
The subsolution corresponds via the following isomorphism to Case 14.

$$\phi : [a3-a9, a4-a10, a5-a11, a13-a13, a17-a17, a9-a5, a11-a7, a12-a3, a2-a8]$$

end

□

Case 14. *Let P be the partially ordered set as visualised in Figure 16. The 3-rd largest element of P can be computed by at most 4 comparisons.*

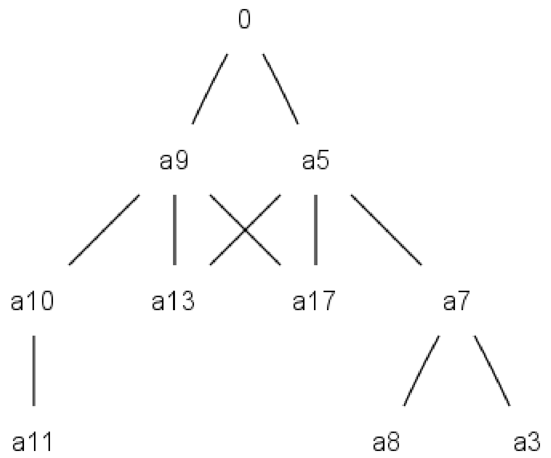

Figure 16: Poset P

Proof. Compare a10 and a7.
if *a10 > a7* **then**
Selecting the 2nd largest element takes at most f2(1,2)=3 further comparisons Kislitsyn (1964).
else
Selecting the 2nd largest element takes at most f2(2,2)=3 further comparisons Kislitsyn (1964).
end

□

Case 15. *Let P be the partially ordered set as visualised in Figure 17. The 3-rd largest element of P can be computed by at most 6 comparisons.*

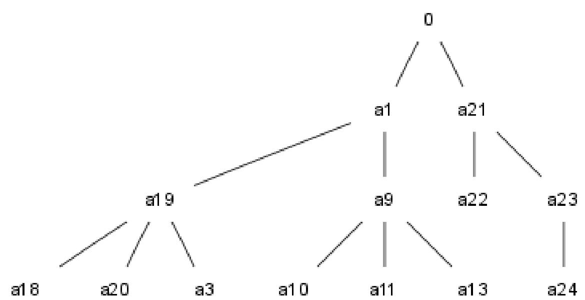

Figure 17: Poset P

Proof. Compare a9 and a21.
if *a9 > a21* **then**
Selecting the 2nd largest element takes at most f2(3,4)=5 further comparisons Kislitsyn (1964).
else
The subsolution corresponds via the following isomorphism to Case 16.

$$\phi : [a1-a9, a19-a11, a18-a12, a20-a13, a3-a17, a9-a10, a21-a3, a22-a2, a23-a4, a24-a5]$$

end

□

Case 16. *Let P be the partially ordered set as visualised in Figure 18. The 3-rd largest element of P can be computed by at most 5 comparisons.*

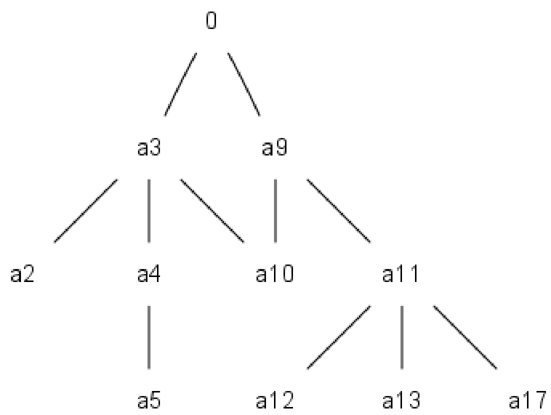

Figure 18: Poset P

Proof. Compare a3 and a9.
if *a3 > a9* **then**
Selecting the 2nd largest element takes at most f2(0,1,2)=4 further comparisons Kislitsyn (1964).
else
Selecting the 2nd largest element takes at most f2(3,3)=4 further comparisons Kislitsyn (1964).
end

□

Case 17. *Let P be the partially ordered set as visualised in Figure 19. The 4-th largest element of P can be computed by at most 7 comparisons.*

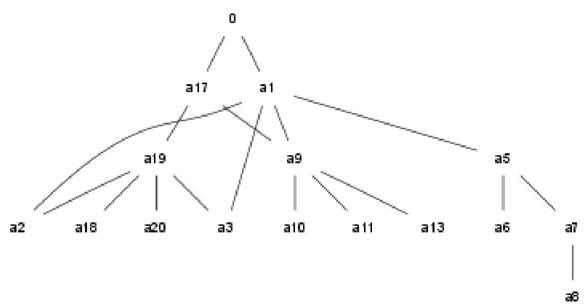

Figure 19: Poset P

Proof. Compare a19 and a1.
if *a19 > a1* **then**
Selecting the 2nd largest element takes at most f2(0,0,4)=6 further comparisons Kislitsyn (1964).
else
The subsolution corresponds via the following isomorphism to Case 18.

$$\phi : [a5-a5, a6-a6, a7-a7, a8-a8, a17-a9, a9-a13, a10-a10, a11-a14, a13-a15, a19-a17, a18-a18, a2-a19, a20-$$

end

□

Case 18. *Let P be the partially ordered set as visualised in Figure 20. The 3-rd largest element of P can be computed by at most 6 comparisons.*

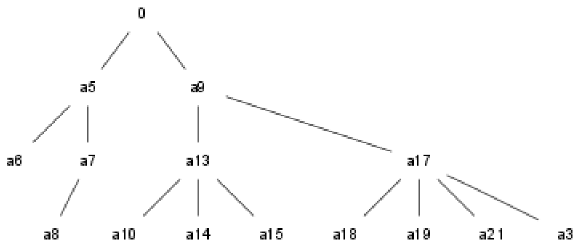

Figure 20: Poset P

Proof. Compare a13 and a5.
if *a13 > a5* **then**
Selecting the 2nd largest element takes at most f2(4,4)=5 further comparisons Kislitsyn (1964).
else
The subsolution corresponds via the following isomorphism to Case 19.

$$\phi : [a9-a9, a17-a17, a18-a18, a19-a19, a21-a21, a3-a3, a13-a10, a5-a5, a6-a6, a7-a7, a8-a8]$$

end

□

Case 19. *Let P be the partially ordered set as visualised in Figure 21. The 3-rd largest element of P can be computed by at most 5 comparisons.*

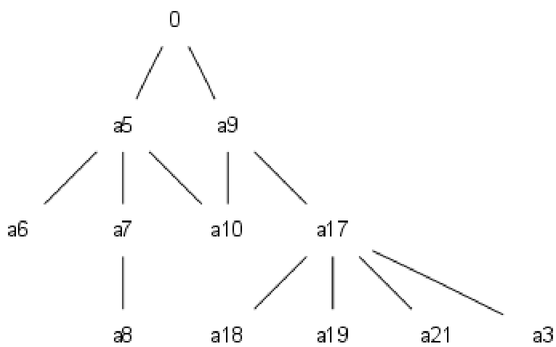

Figure 21: Poset P

Proof. Compare a17 and a5.
if *a17 > a5* **then**
Selecting the 1st largest element takes at most f1(5)=4 further comparisons Knuth (1973).

else
Selecting the 2nd largest element takes at most f2(0,1,2)=4 further comparisons Kislitsyn (1964).
end

□

References

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